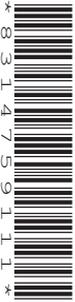# OCR
Oxford Cambridge and RSA

# Friday 22 October 2021 – Afternoon

## A Level Further Mathematics B (MEI)

**Y436/01** Further Pure with Technology

**Time allowed: 1 hour 45 minutes**

## INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. You can use extra paper if you need to, but you must clearly show your candidate number, the centre number and the question numbers.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

## INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

## ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

**1** A family of circles is given by the equation

$$(x - 2\cos a)^2 + (y - 2\sin a)^2 = 1 \; (*)$$

where the parameter $a$ satisfies $0 \leqslant a < 2\pi$.

**(a)** Use a slider (for $a$) to investigate this family of circles. Write down the cartesian equation of the curve which contains the centre of each circle in the family. **[1]**

**(b)** Let $b$ and $c$ be real numbers with $0 \leqslant b < c < \pi$. Find and simplify an expression, in terms of $b$ and $c$, for the distance between the centre of the circle corresponding to $a = b$ and the centre of the circle corresponding to $a = c$. **[2]**

**(c)** Hence, or otherwise, find a condition on $b$ and $c$ for the two circles in part **(b)** to touch. **[2]**

A curve which every member of a family of curves or lines touches tangentially is called an *envelope* of the family.

**(d)** By tracing the family of curves using a slider (for $a$), or otherwise, sketch the envelope of the family (*) in the Printed Answer Booklet. **[2]**

**(e)** Write down the equations of the curves which make up the envelope for this family (*). **[2]**

**2** This question is about the family of straight lines which pass through the points $(0, a)$ and $(1, a^2)$ where the parameter $a$ is any real number.

**(a)** In terms of $a$, find the equation of the straight line which passes through the points $(0, a)$ and $(1, a^2)$. **[2]**

**(b)** Let $b$ and $c$ be distinct real numbers. Given that the straight line corresponding to $a = b$ and the straight line corresponding to $a = c$ are parallel, find $b$ in terms of $c$. **[3]**

**(c)** By tracing the family using a slider (for $a$), or otherwise, sketch the envelope of this family in the Printed Answer Booklet. **[2]**

**(d)** Determine, in the form $y = \mathrm{h}(x)$, the cartesian equation of the envelope for this family. **[5]**

**3** **(a)** **(i)** Create a program which returns the highest common factor of positive integers $m$ and $n$. Write out your program in full in the Printed Answer Booklet. **[3]**

In the rest of this question the highest common factor of positive integers $m$ and $n$ is denoted by $(m, n)$.

**(ii)** Use your program to find $(74333, 89817)$. **[1]**

**(b)** Euler's totient function $\varphi(n)$, where $n$ is a positive integer, is defined to be the number of integers $m$ with $1 \leqslant m \leqslant n$ such that $(m, n) = 1$.
For example $\varphi(6) = 2$ because $(1, 6) = 1$, $(2, 6) = 2$, $(3, 6) = 3$, $(4, 6) = 2$, $(5, 6) = 1$ and $(6, 6) = 6$.

**(i)** Extend your program in **(a)(i)** to create a program which returns $\varphi(n)$ for a given positive integer $n$. **[3]**

**(ii)** Use your program to find $\varphi(128)$ and $\varphi(1000)$. **[2]**

**(iii)** For a positive integer $n$, determine $\varphi(2^n)$ in terms of $n$. **[2]**

**(iv)** For a positive integer $n$, determine $\varphi(10^n)$ in terms of $n$. **[3]**

**(c)** For any positive integer $k$, let $F(k)$ be the number of distinct fractions $\dfrac{m}{n}$ where $0 < m < n \leqslant k$. For example $F(4) = 5$, since there are five fractions which satisfy the required condition, namely $\dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}$.

**(i)** Find $F(5)$ and $F(6)$. **[2]**

**(ii)** Explain why, for any positive integer $l$, $F(l+1) = F(l) + \varphi(l+1)$. **[2]**

**(iii)** Determine $F(100)$. **[2]**

**4** This question concerns the family of differential equations

$$\frac{dy}{dx} = \frac{1-x}{2(x+1)} + a\arctan(y) \quad (x \geqslant 0) \ (*)$$

where $a$ is a constant.

**(a) (i)** Find the solution to (*) in the case $a = 0$ in which $y = 0$ when $x = 0$. **[1]**

 **(ii)** Sketch this solution for $0 \leqslant x \leqslant 5$ in the Printed Answer Booklet. **[1]**

 **(iii)** For this solution, determine the maximum value of $y$ for $0 \leqslant x \leqslant 5$. **[2]**

**(b)** Fig 4.1 and Fig 4.2 show tangent fields for two distinct but unspecified values of $a$. In each case a sketch of the solution curve $y = g(x)$ which passes through the origin is shown for $0 \leqslant x \leqslant 1$.
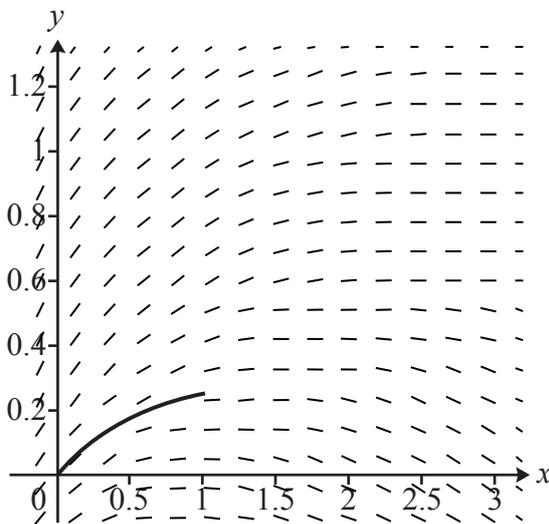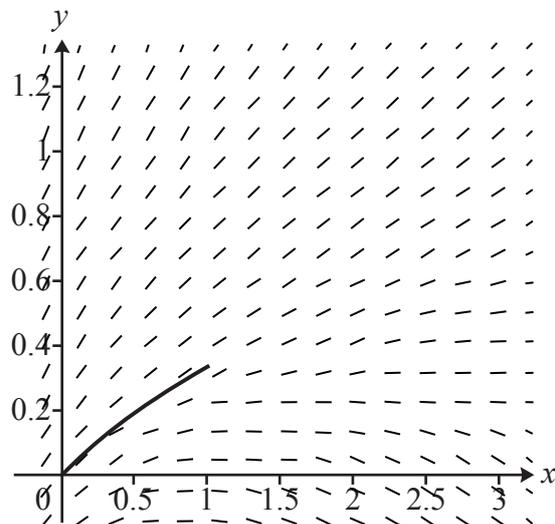


|  |  |
|:---:|:---:|
| **Fig 4.1** | **Fig 4.2** |

 **(i)** For the case in Fig 4.1 suggest a possible value of $a$. **[1]**

 **(ii)** For the case in Fig 4.2 suggest a possible value of $a$. **[1]**

 **(iii)** In each case, continue the sketch of the solution curves for $1 \leqslant x \leqslant 3$ in the Printed Answer Booklet. **[2]**

 **(iv)** State a feature which is present in one of the curves in part **(iii)** but not in the other. **[1]**

**(c) (i)** A modified Euler method for the solution of the differential equation $f(x, y) = \dfrac{dy}{dx}$ is as follows.

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2).$$

Construct a spreadsheet to solve (*), so that the value of $a$ and the value of $h$ can be varied, in the case $x_0 = 0$ and $y_0 = 0$. State the formulae you have used in your spreadsheet. **[4]**

**(ii)** In this part of the question $a = 0.5$. Use your spreadsheet with $h = 0.1$ to approximate the value of $y$ when $x = 5$ for the solution to (*) in which $y = 0$ when $x = 0$. **[1]**

**(iii)** In this part of the question $a = 1$. Use your spreadsheet with $h = 0.1$ to approximate the value of $y$ when $x = 5$ for the solution to (*) in which $y = 0$ when $x = 0$. **[1]**

There is a value $c$ such that

- if $a > c$ then the solution in which $y = 0$ when $x = 0$ increases without bound as $x$ increases from 0

and

- if $a < c$ then the solution in which $y = 0$ when $x = 0$ increases initially but then peaks and decreases as $x$ increases from 0.

**(iv)** Use your spreadsheet to find $c$ correct to 2 decimal places. **[4]**

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